Algebraic Topology: Final exam
May 6, 2004

Attempt any six questions. Each question is worth 20 points. The maximum score is 120

1. Let X be a path-connected, locally path-connected, semi-locally simply-connected space with $\pi_1(X)$ the free group on two generators.
   (a) Show that any connected cover of order two of X is Galois.
   (b) Show that X has exactly three connected covers of order two up to equivalence.
2. The Klein bottle K is the quotient of the square $[0,1] \times [0,1]$ under the identifications $(x,0) \sim (x,1)$ and $(0,y) \sim (1,1-y)$. Compute the homology of K with coefficients in $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$.
3. Let $X = X_1 \cup X_2$ with X_1 and X_2 path-connected, open sets and with $X_1 \cap X_2$ having two path components. Show that $H_1(X) \neq 0$.
4. Let T^2 be the two-dimensional torus and let $f : S^2 \to T^2$ be a map. Show that $f_* : H_2(S^2) \to H_2(T^2)$ is the zero homomorphism.
5. Let $f : T^2 \to T^2$ be an orientation preserving homeomorphism without fixed points. Show that the only eigenvalue of $f_* : H_1(T^2) \to H_1(T^2)$ is 1.
6. Suppose M is a closed, connected, n-dimensional manifold with $H_n(M, \mathbb{Z}/3\mathbb{Z}) = \mathbb{Z}/3\mathbb{Z}$. Show that M is orientable.
7. Let R be a ring with unit. Without using the universal coefficients theorem, prove the following.
   (a) Show that $H^k(S^n; R) = H^k(B^n, S^{n-1}; R)$.
   (b) For $k > 0$, show that $H^k(S^n; R) = H^{k+1}(B^{n+1}, S^n; R)$.
   (c) Show that $H^0(S^0; R) = H^1(B^1, S^0; R) \oplus R$.
   (d) Using the above, compute $H^k(S^n; R)$ for all $k, n \geq 0$